

Quantization in singular real polarizations: Kähler regularization, Maslov correction and pairings

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January 21, 2015

Abstract

We study the Maslov correction to semiclassical states by using a Kähler regularized BKS pairing map from the energy representation to the Schrödinger representation. For general semiclassical states, the existence of this regularization is based on recently found families of Kähler polarizations degenerating to singular real polarizations and corresponding to special geodesic rays in the space of Kähler metrics. In the case of the one-dimensional harmonic oscillator, we show that the correct phases associated with caustic points of the projection of the Lagrangian curves to the configuration space are correctly reproduced.

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1 Introduction

Though Kähler quantization is mathematically better defined, the quantization in (possibly singular) real or mixed polarizations is frequently physically more interesting. This is partly due to the fact that the observables preserving mixed polarizations are likely to be physically more interesting than those preserving a Kähler polarization. In the present paper we continue along the lines proposed in [BFMN, KW2, KMN1, Ki], which motivate the definition of the quantization for real or mixed polarizations via degeneration of quantizations on suitable families of Kähler polarizations.

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While for translation invariant real or mixed polarizations on a symplectic vector space it is easy to construct families of Kähler polarizations degenerating to the given one, for singular polarizations with lower dimensional (thus singular) fibers it is usually not easy to find such Kähler families explicitly (see the no-go Theorem 3.4 and the Conjecture 3.5 below). Building up on previous works [BFMN, KW2, KMN1, MN1], in [MN2] a general strategy is proposed to find families of well behaved polarizations degenerating to a wide class of singular real polarizations corresponding to the level sets of completely integrable systems.

To obtain a direct link between standard half-form corrected quantization and the Maslov phases appearing in the definition of semiclassical states associated with Lagrangian submanifolds, we adopt the following strategy. We consider cases for which the given Lagrangian submanifold is a leaf of a (possibly singular) Lagrangian fibration corresponding to the level sets of a moment map μ of a completely integrable system. Denoting the corresponding real polarization by \mathcal{P}_μ , the proposal of [MN2] corresponds then to obtaining the Hilbert space \mathcal{H}_μ , of quantum states in the polarization \mathcal{P}_μ , as the infinite imaginary time limit $\sqrt{-1}s$ with $s \rightarrow +\infty$, of the family defined by applying the imaginary time flow of the Hamiltonian vector field of the norm square of the moment map, $X_{\|\mu\|^2}$, to the Hilbert space corresponding to a starting Kähler quantization. In order to relate the Schrödinger representation to this Kähler polarization, we consider the Thiemann complexifier method [Th1, Th2] adapted to geometric quantization in [HK1, HK2, KMN2, KMN3, KMN4]. In the toric case, these families of polarizations were first introduced in [BFMN]. These families were also used in [KMN1] to derive the Maslov shift of levels of the Bohr-Sommerfeld leaves. The momentum space quantization T^*K , for compact Lie groups K , was also defined in [KW2] through the infinite imaginary time limit of the flow of the Hamiltonian vector field of the norm squared of the (in general non-abelian) moment map of the action of K on T^*K (see also [KMN2]).

The Maslov correction has been extensively studied (see, for example, [EHHL, Go, GS, MF, Wo, Wu] and references therein). In particular, in [Wu] the holonomy of the natural projectively flat connection along geodesic triangles in the space of Kähler polarizations on \mathbb{R}^{2n} invariant under translations, was shown to yield the triple Maslov index of Kashiwara when the vertices of the triangle approach mutually transverse polarizations at geodesic infinity.

In the present paper, we focus on obtaining semiclassical states associated with real polarizations on \mathbb{R}^{2n} , non-invariant under translations. We are particularly interested in polarizations for which only Kähler regularizations of the second type exist (see section 3 below). We propose a general formalism in section 4.1 and apply it to the harmonic oscillator in section 4.2.

2 Preliminaries

Let (M, ω, I) be a connected Kähler manifold such that $[\frac{\omega}{2\pi\hbar}]$, $\frac{1}{2}c_1(M) \in H^2(M, \mathbb{Z})$ so that the canonical line bundle K_I has I -holomorphic square roots. Let $\sqrt{K_I}$ denote one such square root with Chern connection ∇^I and let us fix a complex line bundle $L \rightarrow M$ with first Chern class $c_1(L) = [\frac{\omega}{2\pi\hbar}]$. We consider on L a connection ∇ with curvature $F_\nabla = -\frac{i}{\hbar}\omega$ and a compatible Hermitian structure h^L . The half-form corrected quantum Hilbert space corresponding to I is then

$$\mathcal{H}_{\mathcal{P}_I} = \overline{\left\{ s \in \Gamma(L \otimes \sqrt{K_I}) : \left(\nabla_{\mathcal{P}_I} \otimes 1 + 1 \otimes \nabla_{\mathcal{P}_I}^I \right) s = 0 \right\}}, \quad (2.1)$$

where \mathcal{P}_I denotes the polarization generated by I -anti-holomorphic vector fields and the bar denotes completion with respect to the inner product defined by

$$\langle \sigma, \sigma' \rangle = \frac{1}{(2\pi\hbar)^n} \int_X h^L(\sigma, \sigma') \frac{\omega^n}{n!}. \quad (2.2)$$

In cases when the canonical bundle K_I is trivial and Ω_I is a global trivializing section we choose as $\sqrt{K_I}$ the trivial square root and denote by $\sqrt{\Omega_I}$ one of the two trivializing sections of $\sqrt{K_I}$ which square to Ω_I .

The half-form corrected prequantization of a function $f \in C^\infty(M)$ is given by

$$\hat{f}^{pQ} = i\hbar \nabla_{X_f} \otimes 1 + f \otimes 1 + i\hbar 1 \otimes \mathcal{L}_{X_f},$$

where X_f denotes the Hamiltonian vector field corresponding to f , or, if a local trivializing section σ of L is given, such that

$$\nabla \sigma = \frac{i}{\hbar} \Theta \sigma, \quad (2.3)$$

where $d\Theta = -\omega$, we obtain, in the local trivialization of L defined by σ ,

$$\hat{f}^{pQ} = i\hbar X_f \otimes 1 - L_f \otimes 1 + i\hbar 1 \otimes \mathcal{L}_{X_f}, \quad (2.4)$$

where $L_f = \Theta(X_f) - f$ is called the Lagrangian of f .

3 Kähler regularizations

In the present section we study regularizations associated with the imaginary time flow of hamiltonians, $h \in C^\infty(M)$, which we call regulators. Depending on the mixed polarization \mathcal{P} we will consider regulators of two types.

Definition 3.1 *\mathcal{P} -regulators of the first type or Thiemann (partial) complexifiers are regulators h for which, there exists a $T > 0$ such that the polarization $\mathcal{P}_{it}^h = e^{it\mathcal{L}_{X_h}}(\mathcal{P})$ exists and is Kähler for $t \in (0, T)$.*

In interesting families of examples, we can then define also a sensible limit of the corresponding Hilbert spaces of polarized quantum states,

$$\mathcal{H}_{\mathcal{P}} := \lim_{t \rightarrow 0} \mathcal{H}_{\mathcal{P}_{it}^h}^h. \quad (3.1)$$

In the examples in [KMN1, KMN2, KMN4], the space of \mathcal{P} -polarized quantum states was already known and the limit actually recovers the correct Hilbert space. Conjecturally, however, one could possibly start with a badly behaved (and hence difficult to quantize directly) polarization \mathcal{P} and define $\mathcal{H}_{\mathcal{P}}$ as a limit of well-behaved quantizations in Kähler polarizations.

Regulators of the first type were introduced by Thiemann in the context of non-perturbative quantum gravity [Th1, Th2] to transform the $SU(2)$ spin connection to the $SL(2, \mathbb{C})$ Ashtekar connection. The prototypical example in finite dimensions is that of the vertical polarization on a cotangent bundle $M = T^*X$ of a compact manifold X . Hall and Kirwin [HK1, HK2] showed, both for the canonical symplectic form ω_c and for a symplectic form modified by a magnetic field, $\omega_b + B$, that the imaginary time flow of the kinetic energy, $h = E_\gamma$, corresponding to a Riemannian metric γ on X defines, at $t = 1$ and on a tubular neighborhood of the zero section, a Kähler structure, which, for $B = 0$, coincides with the adapted Kähler structure introduced by Guillemin-Stenzel [GS1, GS2] and Lempert-Szöke [LS]. In the cases when the Kähler structure extends to T^*X , as is the case of compact Lie groups X with bi-invariant metric, E_γ can be used as a regulator of the first type [KMN2].

Remark 3.2 Regulators of the first type however do not allow to obtain Kähler regularizations of many real polarizations as we show below in Theorem 3.4. See also the Conjecture 3.5. \diamond

Definition 3.3 We call $h \in C^\infty(M)$ a \mathcal{P} -regulator of the second type or (partial) decomplexifier if there exist a polarization \mathcal{P}_0 such that the polarization $\mathcal{P}_{it}^h = e^{it\mathcal{L}_{X_h}}(\mathcal{P}_0)$ exists and is Kähler for $t > 0$ and

$$\lim_{t \rightarrow +\infty} \mathcal{P}_{it}^h = \mathcal{P},$$

in an open, dense subset of M .

Then, as above, one can look for a sensible definition of the space of \mathcal{P} -polarized quantum states by considering the limit

$$\mathcal{H}_{\mathcal{P}} := \lim_{t \rightarrow \infty} \mathcal{H}_{\mathcal{P}_{it}^h}, \quad (3.2)$$

in an appropriate sense.

The need for regulators of the second type comes from the difficulty in finding regulators of the first type for example for real polarizations with compact fibers. In fact, we can prove easily the following result concerning the important case of completely integrable systems on compact manifolds.

Theorem 3.4 Let (M, ω) be a compact real analytic completely integrable system defined by n Hamiltonian functions H_1, \dots, H_n in Poisson involution, with $dH_1 \wedge \dots \wedge dH_n \neq 0$ on an open dense subset of M . Let \mathcal{P} be the real (necessarily singular) polarization with integral leaves corresponding to the level sets of $\mu = (H_1, \dots, H_n)$. Then there can be no real analytic \mathcal{P} -regulator of the first type.

Proof. Recall that \mathcal{P} is pointwise generated by the global Hamiltonian vector fields $X_{H_j}, j = 1, \dots, n$. Suppose that there exists a \mathcal{P} -regulator of the first type, $h \in C^\omega(M)$. From [MN1], it then follows that there exists $\epsilon > 0$ such that $\mathcal{P}_\tau^h = e^{\tau\mathcal{L}_{X_h}}\mathcal{P}$ is, for all $\tau \in \mathbb{C} : |\tau| < \epsilon$, a polarization generated by the (complex) Hamiltonian vector fields of the global functions $H_j^\tau = e^{\tau X_h} H_j$. Then there exists a $\epsilon' \leq \epsilon$ such that \mathcal{P}_{it}^h is Kähler for all $t : 0 < |t| < \epsilon'$ and H_j^{it} are nonconstant global holomorphic functions which contradicts the compactness of M . ■

Conjecture 3.5 We conjecture that do not exist regulators of the first type for singular polarizations \mathcal{P} such that there exist points $x \in M$ for which \mathcal{P}_x is an isotropic non-Lagrangian subspace of $T_x M \otimes \mathbb{C}$.

Fortunately, as shown in [KMN4, MN2], there are regulators of the second type for many of the above examples. They are given by strongly convex functions of the Hamiltonians H_j .

4 Schrödinger semiclassical states and Maslov phases

4.1 Kähler regularized semiclassical states

Let $\mathcal{L} \subset T^*\mathbb{R}^n$ be a compact closed Lagrangian submanifold and consider the Schrödinger representation, that is the prequantum line bundle $L = T^*\mathbb{R}^n \times \mathbb{C}$ with global trivializing section the constant function, with connection $\nabla : \nabla 1 = \frac{i}{\hbar} p dq = \frac{i}{\hbar} \sum_{j=1}^n p_j dq_j$ and

$$\begin{aligned} \mathcal{P}_{Sch} &= \left\langle \frac{\partial}{\partial p} \right\rangle_{\mathbb{C}} = \left\langle \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right\rangle_{\mathbb{C}} \\ \mathcal{H}_{Sch}^Q &= L^2(\mathbb{R}^n, d^n q) \otimes \sqrt{d^n q}. \end{aligned} \quad (4.1)$$

Our goal in the present section will be to use Kähler regularization to construct a semiclassical state $\psi_{\mathcal{L}}$ in the Schrödinger representation that is an approximate eigenvector of a quantum Hamiltonian \hat{h} corresponding to the quantization of a function $h \in C^\infty(T^*\mathbb{R}^n)$ such that

$$\mathcal{L} \subset \{(q, p) \in \mathbb{R}^{2n} : h(q, p) = E\}. \quad (4.2)$$

$\psi_{\mathcal{L}}$ will therefore be an approximate solution of the eigenvalue equation

$$\hat{h}\psi = E\psi. \quad (4.3)$$

Such states have been obtained mainly by using the WKB method of constructing approximate solutions of (4.3) and then imposing the Maslov correction to improve the solution (see, for example, [Wo, GS, BW]). The Maslov correction changes the energy levels (and therefore the set of quantizable Lagrangians) by correcting the Bohr-Sommerfeld quantization conditions and introduces phases in the caustic points of the projection $\pi(\mathcal{L})$, where π is the canonical projection $\pi : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$, $\pi(q, p) = q$.

To construct $\psi_{\mathcal{L}}$ with the help of Kähler regularization, we will consider Kähler regularizations of both the energy and Schrödinger representations, such that they are both deformed, through one-parameter families, to Kähler polarizations. We then use the limit of the BKS pairing map between the Kähler polarizations along these families, to define the pairing map B from the energy representation to the Schrödinger representation. We construct $\hat{\psi}_{\mathcal{L}}$ in the energy representation and define $\psi_{\mathcal{L}} = B(\hat{\psi}_{\mathcal{L}})$ (see (4.13), (4.18)).

For the vertical polarization, $\mathcal{P}_{Sch} = \langle \frac{\partial}{\partial p_j}, j = 1, \dots, n \rangle_{\mathbb{C}}$, there are many Thiemann complexifiers or regulators of the first type. It follows from [KMN2] that any strongly convex function of the momenta is a \mathcal{P}_{Sch} regulator of the first type. Functions of both p and q can also be used as, for example, the Hamiltonians of harmonic oscillators which will be studied in [Es]. Let h_1 denote such a regulator and assume that, eventhough h_1 does not preserve the vertical polarization (otherwise it could not be a \mathcal{P}_{Sch} -regulator of the first type), it has a natural quantization on the Schrödinger representation, \hat{h}_1^{Sch} . Then let

$$\mathcal{P}_{it}^{h_1} = e^{it\mathcal{L}_{x_{h_1}}} \mathcal{P}_{Sch} \quad (4.4)$$

and define the following Kostant–Souriau–Heisenberg (KSH) regularization map for Schrödinger states

$$U_1^{it} = e^{\frac{t}{\hbar}\hat{h}_1^{pQ}} \circ e^{-\frac{t}{\hbar}\hat{h}_1^{Sch}} : \mathcal{H}_{\mathcal{P}_{Sch}} \longrightarrow \mathcal{H}_{\mathcal{P}_{it}^{h_1}}, \quad (4.5)$$

where \hat{h}_1^{pQ} is given by (2.4). In the context of studying the equivalence of quantizations for different polarizations, this map was introduced in [KMN2, KMN3] for the case of cotangent bundles of a compact Lie group, while for toric symplectic manifolds it was considered in [KMN4], and, more generally, it is studied in [MN2]. (For K a compact Lie group, $M = T^*K$ and h_1 the Hamiltonian corresponding to geodesic motion on K for the bi-invariant metric, the KSH map is equivalent to the Hall coherent state transform [Ha1, Ha2], see [KMN2].)

Next, we will need to assume that h is an Hamiltonian in a classically completely integrable system with integrals of motion defining a moment map, $\mu = (H_1, \dots, H_n) : \mathbb{R}^{2n} \cong T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$, such that \mathcal{L} is a corrected Bohr-Sommerfeld fiber. I.e., for some $c_0 \in \mathbb{R}^n$,

$$\mathcal{L} = \mathcal{L}_{c_0} = \{(q, p) \in \mathbb{R}^{2n} : \mu(q, p) = c_0\}. \quad (4.6)$$

and

$$h = F \circ \mu, \quad (4.7)$$

for some $F \in C^\infty(\mathbb{R}^n)$ with $F(c_0) = E$. Let \mathcal{P}_μ denote the real polarization having the level sets $\mathcal{L}_c, c \in \mathbb{R}^n$ as leaves.

Definition 4.1 *We will call the polarization \mathcal{P}_μ the energy polarization and the corresponding quantization on $\mathcal{H}_{\mathcal{P}_\mu}$ the energy quantization.*

A real polarization on \mathbb{R}^{2n} having a compact fiber will typically have singular (lower dimensional) fibers and therefore, due to Conjecture 3.5, we do not expect a Kähler regulator of the first type to exist for \mathcal{P}_μ so that we will need to consider a regulator of the second type for \mathcal{P}_μ . This problem was studied in the toric case in [BFMN, KMN1] and in general in [MN2] and we will review now some of the results.

Let us assume that the level sets \mathcal{L}_c are compact for noncritical values $c \in \mathbb{R}^n$ and that a function $G : \mathbb{R}^n \rightarrow \mathbb{R}$ exists such that $h_2 = G \circ \mu$ is strongly convex as a function of all action variables on equivariant neighborhoods of all regular fibers. This, plus some technical assumptions on the Fourier coefficients of local holomorphic functions for some initial complex structure J_0 , imply that the polarization

$$\mathcal{P}_{it}^{h_2} = e^{it\mathcal{L}_{X_{h_2}}} \mathcal{P}_0, \quad (4.8)$$

where \mathcal{P}_0 is the polarization associated with J_0 , converges to \mathcal{P}_μ as $t \rightarrow \infty$,

$$\lim_{t \rightarrow +\infty} \mathcal{P}_{it}^{h_2} = \mathcal{P}_\mu,$$

in an appropriate (weak) sense [BFMN, MN2]. We will also assume that there are local J_0 -holomorphic coordinates, $\{u_j\}_{j=1,\dots,n}$, such that pointwise, on a neighborhood of every point, one has

$$\lim_{t \rightarrow \infty} \beta(t) du_1^{(it)} \wedge \dots \wedge du_n^{(it)} = dH_1 \wedge \dots \wedge dH_n, \quad (4.9)$$

for some smooth, positive function $\beta \in C^\infty((0, \infty))$, where $u_j^{(it)} = e^{itX_{h_2}}(u_j)$. Consider the following modification of the KSH map, introduced in [MN2],

$$U_2^{it} = e^{-\frac{t}{\hbar} \hat{h}_2^\mu} \circ e^{\frac{t}{\hbar} \hat{h}_2^{pQ}}, \quad (4.10)$$

where \hat{h}_2^{pQ} is defined in (2.4) and \hat{h}_2^μ is the following self adjoint operator

$$\hat{h}_2^\mu \left(f \otimes \sqrt{du_1^{(it)} \wedge \dots \wedge du_n^{(it)}} \right) = G(\tilde{H}_1^{pQ}, \dots, \tilde{H}_n^{pQ})(f) \otimes \sqrt{du_1^{(it)} \wedge \dots \wedge du_n^{(it)}}, \quad (4.11)$$

densely defined on $L^2(\mathbb{R}^{2n}) \otimes \sqrt{du_1^{(it)} \wedge \dots \wedge du_n^{(it)}}$, where \tilde{H}_j^{pQ} denotes the prequantization of H_j without the half-form correction, $\tilde{H}_j^{pQ} = i\hbar X_{H_j} - L_{H_j}$ (compare with (2.4)). The operator U_2^{it} in (4.10) maps, for all $t > 0$, $\mathcal{H}_{\mathcal{P}_0}$ to, in general non-polarized, subspaces of

$$L^2(\mathbb{R}^{2n}) \otimes \sqrt{du_1^{(it)} \wedge \dots \wedge du_n^{(it)}}.$$

We will further assume that the following limit, for every $\psi \in \mathcal{H}_{\mathcal{P}_0}$,

$$U_2^{i\infty}(\psi) = \lim_{t \rightarrow \infty} U_2^{it}(\psi) \quad (4.12)$$

exists and the resulting map, $U_2^{i\infty} : \mathcal{H}_{\mathcal{P}_0} \rightarrow \mathcal{H}_{\mathcal{P}_\mu}$, is an isomorphism onto the space of polarized Dirac delta distributions supported on Maslov corrected Bohr-Sommerfeld Lagrangian leaves (see [BFMN, KMN1, MN2]). Then the distributional section

$$\hat{\psi}_{\mathcal{L}_{c_0}}(H, \theta) = \delta(H - c_0) e^{\frac{i}{\hbar} \tilde{c}_0 \cdot \theta} \otimes \sqrt{d^n H} \in \mathcal{H}_{\mathcal{P}_\mu}, \quad (4.13)$$

where (H, θ) are local action-angle coordinates, the phase factor $\tilde{c}_0 \in \hbar\mathbb{Z}^n$ corresponds to the uncorrected Bohr-Sommerfeld leaf, $H = \tilde{c}$, (compare with (4.31) for the harmonic oscillator), is the image of a uniquely defined section $\tilde{\psi}_{\mathcal{L}} \in \mathcal{H}_{\mathcal{P}_0}$,

$$\hat{\psi}_{\mathcal{L}} = U_2^{i\infty}(\tilde{\psi}_{\mathcal{L}}). \quad (4.14)$$

Summarizing, our proposal to use Kähler regularization to construct semiclassical solutions of (4.3) can be divided in the following steps.

- 1) *Choice of regulators h_1, h_2 and construction of the KSH maps in U_1^{it} in (4.5) and U_2^{it} in (4.10):*
 - i) Choose the Thiemann complexifier or regulator of first type, h_1 , for the Schrödinger polarization and define the one-parameter family of Kähler polarizations (4.4) and Kähler regularizations of the Schrödinger representation (4.5). A standard choice is the free particle Hamiltonian, $h_1(q, p) = \frac{1}{2}\|p\|^2$, for which $\mathcal{P}_{it}^{h_1} = \langle X_{z_1^{(it)}}, \dots, X_{z_n^{(it)}} \rangle_{\mathbb{C}}$, where $z^{(it)} = q + itp$. Another interesting possibility, studied in [Es], consists in using the regulator of second type used for the polarization \mathcal{P}_μ also as Thiemann complexifier, i.e. $h_1 = h_2 = h$. Find U_1^{it} in (4.5).
 - ii) Choose a \mathcal{P}_μ -regulator of second type, h_2 , for the energy representation of Definition 4.1, define the one-parameter family of Kähler polarizations (4.8) and the Kähler regularized Hilbert space as the image, $U_2^{it}(\mathcal{H}_{\mathcal{P}_0})$, of (4.10), for large t , leading to (4.12), (4.13) and (4.14). As mentioned above, from [BFMN, KMN1, KMN4, MN2], it follows that h_2 should be a function on \mathbb{R}^{2n} that is strongly convex in the action coordinates.

- 2) *BKS pairing between the Schrödinger representation and the energy representation of Definition 4.1:*

For the states $\psi_1 \in \mathcal{H}_{Sch}$ and $\psi_2 \in \mathcal{H}_\mu$ define their time (t_1, t_2) regularized BKS pairing as

$$\langle \psi_1, \psi_2 \rangle^{t_1, t_2} = \langle U_1^{it_1}(\psi_1), U_2^{it_2}(\tilde{\psi}_2) \rangle_{BKS}, \quad (4.15)$$

where $\tilde{\psi}_2$ is the state in $\mathcal{H}_{\mathcal{P}_0}$ mapped to ψ_2 by $U_2^{i\infty}$, that is $\tilde{\psi}_2 = (U_2^{i\infty})^{-1}(\psi_2)$ and the pairing in the right hand side is the usual half-form corrected BKS pairing of geometric quantization. The BKS pairing between states in \mathcal{H}_{Sch} and $\mathcal{H}_{\mathcal{P}_\mu}$ is then defined by the following limit of Kähler regularized pairings

$$\langle \psi_1, \psi_2 \rangle_{BKS} = \lim_{(t_1, t_2) \rightarrow (0, \infty)} \langle \psi_1, \psi_2 \rangle^{t_1, t_2}, \quad (4.16)$$

in case the limit exists $\forall \psi_1 \in \mathcal{H}_{Sch}, \psi_2 \in \mathcal{H}_{\mathcal{P}_\mu}$.

- 3) *Definition of the semiclassical state $\psi_{\mathcal{L}}$:*

If the limit in (4.16) exists and is continuous on $\mathcal{H}_{Sch} \times \mathcal{H}_{\mathcal{P}_\mu}$, it defines a pairing map

$$B : \mathcal{H}_{\mathcal{P}_\mu} \rightarrow \mathcal{H}_{Sch}. \quad (4.17)$$

The semiclassical state corresponding to \mathcal{L} via the Kähler regularization of the pairing is then defined to be

$$\psi_{\mathcal{L}} = B(\hat{\psi}_{\mathcal{L}}), \quad (4.18)$$

where $\hat{\psi}_{\mathcal{L}}$ is the state defined in (4.13). So, $\psi_{\mathcal{L}}$ is the unique state in \mathcal{H}_{Sch} such that

$$\langle \psi, \psi_{\mathcal{L}} \rangle_{\mathcal{H}_{Sch}} = \langle \psi, \hat{\psi}_{\mathcal{L}} \rangle_{BKS}, \forall \psi \in \mathcal{H}_{Sch}.$$

Remark 4.2 The pairing map B and thus the semiclassical states, could in principle depend on the Kähler regularizations, though we would expect to obtain the same result for generic choices.

◇

4.2 Maslov phases for the harmonic oscillator

Let us illustrate the general method of the previous section in the case of the one-dimensional harmonic oscillator. We have $n = 1, M = \mathbb{R}^2, \omega = dq \wedge dp, \Theta = pdq, L = \mathbb{R}^2 \times \mathbb{C}, \nabla 1 = \frac{i}{\hbar} \Theta$ and $h(q, p) = \mu = H = \frac{1}{2}(p^2 + q^2)$. As in (4.1), we have

$$\mathcal{H}_{Sch} = L^2(\mathbb{R}, dq) \otimes \sqrt{dq}.$$

For $\mathcal{H}_{\mathcal{P}_\mu}$, since h is the action variable for the standard toric structure in \mathbb{R}^2 , we know from [KMN1] that

$$\mathcal{H}_{\mathcal{P}_\mu} = \langle \delta(h - \hbar(m + \frac{1}{2})) e^{-\frac{i}{2\hbar} pq} e^{im\theta}, m \in \mathbb{N}_0 \rangle_{\mathbb{C}} \otimes \sqrt{dh}. \quad (4.19)$$

Let us now follow the three steps described in the previous section, in this example.

1) *Choice of regulators h_1, h_2 and construction of U_1^{it} in (4.5) and U_2^{it} in (4.10):*

i) Let us choose the standard Thiemann complexifier for the Schrödinger representation, $h_1(q, p) = \frac{1}{2}p^2$. Then

$$\begin{aligned} \mathcal{P}_{it}^{h_1} &= e^{it\mathcal{L}_{X_{h_1}}} \mathcal{P}_{Sch} = e^{it\mathcal{L}_{X_{h_1}}} \langle X_q \rangle_{\mathbb{C}} = \langle X_{e^{itX_{h_1}}(q)} \rangle_{\mathbb{C}} = \\ &= \langle X_{q+itp} \rangle_{\mathbb{C}} = \langle X_{z^{(it)}} \rangle_{\mathbb{C}} \end{aligned}$$

where $z^{(it)} = q + itp$, defines a Kähler polarization for all $t > 0$, confirming that h_1 is a \mathcal{P}_{Sch} -regulator of the first type. This is the simplest example of imaginary time geodesic flows starting at the Schrödinger polarization and leading (at time $= \sqrt{-1}$) to adapted Kähler structures on tubular neighbourhoods (the whole $T^*\mathbb{R}$ in the present case) of the zero section of cotangent bundles of compact Riemannian manifolds [HK1]. The equation for $\mathcal{P}_{it}^{h_1}$ -polarized sections of L reads,

$$\nabla_{X_{z^{(it)}}} \psi = 0 \Leftrightarrow \left(-\frac{\partial}{\partial p} + it \frac{\partial}{\partial q} - p \frac{t}{\hbar} \right) \psi(q, p) = \left(2it \frac{\partial}{\partial \bar{z}^{it}} - p \frac{t}{\hbar} \right) \psi(q, p) = 0,$$

and therefore

$$\mathcal{H}_{\mathcal{P}_{it}^{h_1}} = \left\{ f(z^{(it)}) e^{-\frac{t}{2\hbar} p^2} \otimes \sqrt{dz^{(it)}} : \int_{\mathbb{R}^2} |f(z^{(it)})|^2 e^{-\frac{t}{\hbar} p^2} dq dp < \infty \right\}, \quad (4.20)$$

where f is $\mathcal{P}_{it}^{h_1}$ -holomorphic.

Let us now obtain the corresponding Kähler regularization maps, U_1^{it} . From (2.4) we obtain

$$\hat{p}^{pQ} = i\hbar \left(\frac{\partial}{\partial q} \otimes 1 + 1 \otimes \mathcal{L}_{\frac{\partial}{\partial q}} \right) \text{ and } \hat{h}_1^{pQ} = i\hbar \left(p \frac{\partial}{\partial q} \otimes 1 + 1 \otimes \mathcal{L}_{p \frac{\partial}{\partial q}} \right) - \frac{p^2}{2} \otimes 1. \quad (4.21)$$

We see that, due to the fact that p preserves the Schrödinger polarization, \hat{p}^{pQ} acts on the Schrödinger representation. We can then define

$$\hat{h}_1^{Sch} = \frac{(\hat{p}^{pQ})^2}{2} |_{\mathcal{H}_{Sch}^Q} = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} \otimes 1. \quad (4.22)$$

It is convenient to act with U_1^{it} on $\psi \in \mathcal{H}_{Sch}$, written in the form

$$\psi(q) \otimes \sqrt{dq} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{i}{\hbar} p_0 q} \tilde{\psi}(p_0) dp_0 \otimes \sqrt{dq}.$$

From (4.5), (4.21) and (4.22) we obtain for $U_1^{it}(\psi) \in \mathcal{H}_{\mathcal{P}_{it}^{h_1}}$,

$$U_1^{it}(\psi)(q, p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{i}{\hbar} p_0 q} e^{-\frac{t}{2\hbar}(p+p_0)^2} \tilde{\psi}(p_0) dp_0 \otimes \sqrt{dz^{(it)}}. \quad (4.23)$$

- ii) For the choice of h_2 notice that $h = H = \mu$ is an action coordinate in $\mathbb{R}^2 \setminus \{0\}$ generating a global S^1 action. The angle coordinate is the polar angle $\theta = \arctan(p/q)$. We can therefore choose as h_2 a strongly convex function of $H = h$, e.g.

$$h_2(q, p) = \frac{1}{2} H^2 = \frac{1}{8} (p^2 + q^2)^2.$$

Let $w = z^{(i)} = q + ip = \sqrt{2\hbar} e^{i\theta}$ and choose as starting polarization, the S^1 -invariant toric polarization $\mathcal{P}_0 = \langle X_w \rangle_{\mathbb{C}} = \langle \frac{\partial}{\partial \bar{w}} \rangle_{\mathbb{C}}$. We have $X_{h_2} = -\hbar \frac{\partial}{\partial \theta}$ and therefore the one-parameter family of polarizations obtained by flowing with this vector field in imaginary time is also toric and a simple particular example of those studied in [BFMN] and [KMNI]

$$\begin{aligned} \mathcal{P}_{it}^{h_2} &= e^{it\mathcal{L}_{X_{h_2}}} \mathcal{P}_0 = e^{it\mathcal{L}_{X_{h_2}}} \langle X_w \rangle_{\mathbb{C}} = \langle X_{e^{-it\hbar} \frac{\partial}{\partial \theta} w} \rangle_{\mathbb{C}} = \\ &= \langle X_{w^{(it)}} \rangle_{\mathbb{C}}, \end{aligned}$$

where

$$\frac{w^{(it)}}{\sqrt{2}} = \sqrt{\hbar} e^{th} e^{i\theta} = e^{\frac{1}{2} \log(\hbar) + th + i\theta} = e^{\frac{dg}{dh} + i\theta}, \quad (4.24)$$

and $g(h) = \frac{1}{2} h \log(h) - \frac{h}{2} + t \frac{h^2}{2}$ is the strongly convex toric symplectic potential. (See e.g. [BFMN]). A local J_0 -holomorphic coordinate satisfying (4.9) is $u = \log(w/\sqrt{2}) = th + \frac{1}{2} \log(h) + i\theta$ with $\beta(t) = 1/t$. To find the Hilbert space it will be convenient to use the S^1 -invariant trivializing section (we will return to the trivialization defined by $\sigma(q, p) = 1$ when we calculate the pairing of states of different polarizations) $\tilde{\sigma} = e^{-\frac{i}{2\hbar} qp} \sigma$ so that

$$\nabla \tilde{\sigma} = -\frac{i}{\hbar} h d\theta \tilde{\sigma}.$$

The equation for $\mathcal{P}_{it}^{h_2}$ -polarized sections then reads

$$\nabla_{\frac{\partial}{\partial u}} \psi = 0 \Leftrightarrow \left(\frac{\partial}{\partial v} + i \frac{\partial}{\partial \theta} + \frac{h}{\hbar} \right) \tilde{\psi}(q, p) = 0. \quad (4.25)$$

Since g is strictly convex and $v = \frac{dg}{dh}$, the inverse Legendre map is $h = \frac{dk}{dv}$, where $k(v) = h(v)v - g(h(v))$ is the Kähler potential. We see that the solutions of (4.25) are given by

$$\tilde{\psi}(q, p) = f(w^{(it)}) e^{-\frac{k}{\hbar}} = f(w^{(it)}) e^{-\frac{1}{2\hbar}(th^2 + h)},$$

where f is an arbitrary $\mathcal{P}_{it}^{h_2}$ -holomorphic function. Therefore, we have, in the trivialization defined by $\tilde{\sigma}$,

$$\tilde{\mathcal{H}}_{\mathcal{P}_{it}^{h_2}} = \left\{ f(w^{(it)}) e^{-\frac{1}{2\hbar}(th^2 + h)} \otimes \sqrt{dw^{(it)}} : \int_{\mathbb{R}^+ \times S^1} |f(w^{(it)})|^2 e^{-\frac{1}{\hbar}(th^2 + h)} \sqrt{g''(h)} dh d\theta < \infty \right\}, \quad (4.26)$$

Let us now obtain $U_2^{(it)}$ in (4.10). From (2.4) we obtain

$$\hat{h}^{pQ} = -i\hbar \left(\frac{\partial}{\partial \theta} \otimes 1 + 1 \otimes \mathcal{L}_{\frac{\partial}{\partial \theta}} \right) \text{ and } \hat{h}_2^{pQ} = -i\hbar \left(h \frac{\partial}{\partial \theta} \otimes 1 + 1 \otimes \mathcal{L}_{h \frac{\partial}{\partial \theta}} \right) - \frac{h^2}{2} \otimes 1. \quad (4.27)$$

The sections in the monomial basis of $\tilde{\mathcal{H}}_{\mathcal{P}_{it}^{h_2}}$,

$$\begin{aligned} \varphi_m^{(it)} &= a_m (w^{(it)})^m e^{-\frac{1}{2\hbar}(th^2+h)} \otimes \sqrt{dw^{(it)}} = \\ &= a_m 2^{\frac{m}{2}+\frac{1}{4}} h^{\frac{m}{2}+\frac{1}{4}} e^{-\frac{h}{2\hbar}} e^{-\frac{t}{2\hbar}((h-h(m+\frac{1}{2}))^2)} e^{\frac{th}{2}(m+\frac{1}{2})^2} e^{i(m+1/2)\theta} \sqrt{du^{(it)}}, \quad m \in \mathbb{N}_0, \end{aligned} \quad (4.28)$$

where $a_m = 2^{-\frac{m}{2}-\frac{1}{4}} (2\pi\hbar)^{-\frac{1}{2}} (\hbar(m+\frac{1}{2}))^{-\frac{m}{2}-\frac{1}{4}} e^{\frac{m}{2}+\frac{1}{2}}$, form an orthogonal basis of eigensections of \hat{h}^{pQ} and of \hat{h}_2^μ (see (4.11)), with

$$\hat{h}_2^\mu(\varphi_m^{(it)}) = \frac{1}{2} \left(\hbar(m+\frac{1}{2}) \right)^2 \varphi_m^{(it)}. \quad (4.29)$$

The constants a_m in (4.28) are chosen to have $\varphi_m^{(0)} = \tilde{\psi}_{\mathcal{L}_m}$ (see (4.14)). In this case \hat{h}_2^μ acts on the space of polarized sections because h preserves the polarizations $\mathcal{P}_{it}^{h_2}$ for every $t \in [0, \infty)$. Eventhough h_2 itself does not preserve the polarization the operator \hat{h}_2^μ is defined through \hat{h} in (4.11) and therefore has a well defined action on the Hilbert spaces $\mathcal{H}_{\mathcal{P}_{it}^{h_2}}$. Note that (4.28) is a local expression of a global $\mathcal{P}_{it}^{h_2}$ -holomorphic section of the half-form corrected prequantum bundle in spite of the factor of $e^{\frac{i}{2}\theta}$. In fact, as explained in Section 3 and in the Appendix of [KMN1], this factor gets canceled against a similar factor arising from the fact that $du^{(it)}$ is a meromorphic section of the canonical bundle having a pole of order 1 at the origin. (Or, equivalently, (4.28) is written in a frame of the half-form corrected prequantum bundle which has a square root ramification divisor at the origin.)

To the Lagrangian cycles

$$\mathcal{L}_m = \{(q, p) \in \mathbb{R}^2 : q^2 + p^2 = \hbar(2m+1)\}, \quad m \in \mathbb{N}_0,$$

as in (4.14), there will then correspond the state $\tilde{\psi}_{\mathcal{L}_m} = \varphi_m^{(0)}$, as shown below.

From (4.27) and (4.28) we also obtain that $e^{\frac{t}{\hbar} \hat{h}_2^{pQ}}(\varphi_m^{(0)}) = \varphi_m^{(it)}$ and therefore

$$\begin{aligned} U_2^{it}(\varphi_m^{(0)}) &= e^{-\frac{t}{\hbar} \hat{h}_2^\mu} \circ e^{\frac{t}{\hbar} \hat{h}_2^{pQ}}(\varphi_m^{(0)}) = \\ &= a_m 2^{\frac{m}{2}+\frac{1}{4}} h^{\frac{m}{2}+\frac{1}{4}} e^{-\frac{h}{2\hbar}} e^{-\frac{t}{2\hbar}((h-h(m+\frac{1}{2}))^2)} e^{i(m+1/2)\theta} \sqrt{(1/2h+t)dh+id\theta}. \end{aligned} \quad (4.30)$$

When taking the limit $t \rightarrow +\infty$, following [KMN1] and taking care of the fact that $\sqrt{du^{(it)}}$ effectively carries a factor of $e^{-\frac{i}{2}\theta}$ as remarked above, we obtain

$$\hat{\psi}_{\mathcal{L}_m} = \lim_{t \rightarrow \infty} U_2^{it}(\varphi_m^{(0)}) = \delta(h - \hbar(m+\frac{1}{2})) e^{im\theta} \sqrt{dh}, \quad m \in \mathbb{N}_0 \quad (4.31)$$

so that indeed, $\tilde{\psi}_{\mathcal{L}_m} = \varphi_m^{(0)}$.

2) *BKS pairing between the Schrödinger representation and the energy representation:*

Proposition 4.3 For the holomorphic forms in (4.23) with $t = t_1$ and (4.28) with $t = t_2$, we obtain

$$\left(\frac{i}{2}\right) dz^{(it_1)} \wedge \left(\left(\frac{1}{2h} + t_2\right) dh - id\theta\right) = \left(\frac{i}{2}\right) \left(\frac{(p-iq)(1+t_1)}{p^2+q^2} + t_2(p-it_1q)\right) \omega. \quad (4.32)$$

Proof. The result follows directly from $z^{(it_1)} = q + it_1p$ and (4.24) with $t = t_2$. ■

Proposition 4.4 The pairing in (4.16) for the harmonic oscillator is given by

$$\begin{aligned} \langle \psi, \hat{\psi}_{\mathcal{L}_m} \rangle_{BKS} &= \lim_{(t_1, t_2) \rightarrow (0, +\infty)} \langle U_1^{it}(\psi)(q, p), e^{ipq/2h} U_2^{(it)}(\varphi_m^{(0)}) \rangle_{BKS} = \\ &= \sqrt{\frac{i}{2}} \int_{\mathbb{R}^2} \psi(q) e^{-ipq/2h} e^{-im\theta} \sqrt{p} \delta\left(h - \hbar(m + \frac{1}{2})\right) dq dp, \end{aligned} \quad (4.33)$$

where $\psi \in \mathcal{H}_{Sch}$.

Proof. From (4.23), (4.2) and (4.32) we obtain the BKS pairing

$$\begin{aligned} &\langle U_1^{it}(\psi)(q, p), e^{ipq/2h} U_2^{(it)}(\varphi_m^{(0)}) \rangle_{BKS} = \\ &= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{i}{2}} a_m 2^{\frac{m}{2} + \frac{1}{4}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{\frac{i}{\hbar} p_0 q} e^{-\frac{t_1}{2h}(p+p_0)^2} \tilde{\psi}(p_0) h^{\frac{m}{2} + \frac{1}{4}} e^{-ipq/2h} e^{-\frac{h}{2h}} e^{-\frac{t_2}{2h}((h-\hbar(m+\frac{1}{2}))^2)} e^{-im\theta} \\ &\quad \cdot \left(\frac{(p-iq)(1+t_1)}{p^2+q^2} + t_2(p-it_1q)\right)^{\frac{1}{2}} dp_0 dq dp. \end{aligned}$$

Due to the gaussians, the integrals are convergent and bounded and, therefore, the limit $(t_1, t_2) \rightarrow (0, +\infty)$ exists and can be taken inside the integral.

Taking the limit $t_2 \rightarrow +\infty$ gives,

$$\begin{aligned} \lim_{t_2 \rightarrow +\infty} \langle U_1^{it}(\psi)(q, p), e^{ipq/2h} U_2^{(it)}(\varphi_m^{(0)}) \rangle_{BKS} &= \sqrt{\hbar} a_m 2^{\frac{m}{2} + \frac{1}{4}} \sqrt{\frac{i}{2}} \left(\hbar(m + \frac{1}{2})\right)^{\frac{m}{2} + \frac{1}{4}} e^{-(\frac{m}{2} + \frac{1}{4})} \\ &\cdot \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{\frac{i}{\hbar} p_0 q} e^{-\frac{t_1}{2h}(p+p_0)^2} \tilde{\psi}(p_0) e^{-ipq/2h} e^{-im\theta} (p-it_1q)^{\frac{1}{2}} \delta\left(h - \hbar(m + \frac{1}{2})\right) dp_0 dq dp. \end{aligned}$$

The limit $t_1 \rightarrow 0$ and the integration in dp_0 then produces the result. ■

3) The semiclassical state $\psi_{\mathcal{L}}$:

From above we obtain the following

Proposition 4.5 Let $\mathcal{L}_m, m \in \mathbb{N}_0$, be the Lagrangian cycles where $h = \hbar(m + \frac{1}{2})$, as above. The pairing map in (4.17) and (4.18) for the harmonic oscillator reads

$$B : \mathcal{H}_{\mathcal{P}_\mu} \rightarrow \mathcal{H}_{Sch} \quad (4.34)$$

$$\psi_{\mathcal{L}_m}(q) = B(\hat{\psi}_{\mathcal{L}_m})(q) = \psi_{\mathcal{L}_m}^+(q) + \psi_{\mathcal{L}_m}^-(q),$$

such that $\psi_{\mathcal{L}_m}^+, \psi_{\mathcal{L}_m}^-$ have support in $[-\hbar(2m+1), \hbar(2m+1)]$ where they are given by

$$\psi_{\mathcal{L}_m}^+(q) = \sqrt{\frac{i}{2}} (\hbar(2m+1) - q^2)^{-\frac{1}{4}} e^{\frac{i}{2\hbar} q \sqrt{\hbar(2m+1) - q^2}} e^{im \arctan \frac{\sqrt{\hbar(2m+1) - q^2}}{q}} \otimes \sqrt{dq}$$

and

$$\psi_{\mathcal{L}_m}^-(q) = \sqrt{\frac{i}{2}} e^{i\frac{\pi}{2}} (\hbar(2m+1) - q^2)^{-\frac{1}{4}} e^{\frac{i}{2\hbar} q \sqrt{\hbar(2m+1) - q^2}} e^{-im \arctan \frac{\sqrt{\hbar(2m+1) - q^2}}{q}} \otimes \sqrt{dq}. \quad (4.35)$$

Proof. Let ψ in (4.33) be a continuous function in $L^2(\mathbb{R})$. Then, since the inner product in \mathcal{H}_{Sch} is given by integration in q , we obtain from (4.17), (4.18) and (4.33),

$$\psi_{\mathcal{L}_m}(q) = B(\hat{\psi}_{\mathcal{L}_m})(q) = \psi_{\mathcal{L}_m}^+(q) + \psi_{\mathcal{L}_m}^-(q), \quad (4.36)$$

where

$$\psi_{\mathcal{L}_m}^+(q) = \sqrt{\frac{i}{2}} \int_0^{+\sqrt{2\hbar(m+\frac{1}{2})}} e^{-i\frac{pq}{2\hbar}} e^{im \arctan \frac{p}{q}} \sqrt{p} \delta\left(\frac{p^2 + q^2}{2} - \hbar(m + \frac{1}{2})\right) dp \otimes \sqrt{dq} \quad (4.37)$$

and

$$\psi_{\mathcal{L}_m}^-(q) = \sqrt{\frac{i}{2}} e^{i\frac{\pi}{2}} \int_{-\sqrt{2\hbar(m+\frac{1}{2})}}^0 e^{-i\frac{pq}{2\hbar}} e^{im \arctan \frac{p}{q}} \sqrt{-p} \delta\left(\frac{p^2 + q^2}{2} - \hbar(m + \frac{1}{2})\right) dp \otimes \sqrt{dq} \quad (4.38)$$

and the result follows. ■

We observe that $\psi_{\mathcal{L}_m}$, which is supported in the “classically allowed region” $[-\hbar(2m+1), \hbar(2m+1)]$, contains two contributions, weighted with a relative (Maslov) phase $e^{i\frac{\pi}{2}}$ which arises at the caustic points $q = \pm\sqrt{2\hbar(m+\frac{1}{2})}, p = 0$, for the projection of \mathcal{L}_m onto the q -axis. Moreover, by explicitly evaluating $\int pdq$ with $(q, p) \in \mathcal{L}_m$, we see that $\psi_{\mathcal{L}_m}$ (4.35) has the form of the usual WKB wave function in the classically allowed region. (See, for example, Chapter 7 of [Me].)

Acknowledgements: The authors were partially supported by FCT/Portugal through the projects PEst-OE/EEI/LA009/2013, EXCL/MAT-GEO/0222/2012, PTDC/MAT/-119689/2010, PTDC/MAT/1177762/2010. The first author was supported by the FCT fellowship SFRH/BPD/77123/2011 and by a postdoctoral fellowship of the project PTDC/MAT/120411/2010. The second author thanks S. Wu for useful discussions and is thankful for generous support from the Emerging Field Project on Quantum Geometry from Erlangen–Nürnberg University.

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